HW5. Ross §13. 3, 4, 6, 7

13.3 Let B be the set of bounded seq $x = (x_1, x_2, \cdots)$

define $d(x,y) = \sup\{|x_i - y_i|, i=1,2,\cdots\}$.

(a) Show that d is a metric

(b) Does $d = \sum_{j=1}^{\infty} |x_j - y_j|$ define a metric for B?

Sol: (a) Since (x_n) and (y_n) are bounded, thus $(x_n - y_n)$ is a bounded seq, hence $\sup\{|x_n - y_n|\}$ exists in $\mathbb{R}$.

Hence $d(x,y) \in \mathbb{R}$, $\forall (x_n)\ (y_n)$ in B. The axiom (1) (2) for metric is easy to verify. We now check triangle inequality. $\forall (x_n), (y_n), (z_n)$ bounded seq, we need to show that

$$\sup |x_n - y_n| + \sup |y_n - z_n| \geq \sup |x_n - z_n|.$$

Indeed, $|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n|$.

Hence, $\sup |x_n - z_n| \leq \sup |x_n - y_n| + \sup |y_n - z_n|$.

(b). No, $d^*(x,y)$ might be ∞. i.e. $(x_n) = 1$, $(y_n) = 0$

#13.4 From the definition of open set, prove that

(1) union of arbitrary open set is open

(2) intersection of finitely many open is open

Pf: (1) Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets.

If $p \in \bigcup_\alpha U_\alpha$, then $\exists\ \alpha_0 \in A$, s.t. $p \in U_{\alpha_0}$.

Hence, $\exists \delta > 0$, $B_\delta(p) \subset U_{\alpha_0}$, since U_{α_0} is open.

$\therefore$ $B_\delta(p) \subset U_{\alpha_0} \subset \bigcup_\alpha U_{\alpha_0}$.

(2) If $p \in \bigcap_{i=1}^{N} U_i$, U_i open, then $\exists\, \delta_i > 0$ $\forall i \in \{1, \cdots, N\}$

s.t. $B_{\delta_i}(p) \subset U_i$. Let $\delta = \min\{\delta_i\}$. Then

$B_\delta(p) \subset U_i$, $\forall i \Rightarrow B_\delta(p) \subset \bigcap_{i=1}^{N} U_i$

#13.7 **13.9 Proposition.**

Let E be a subset of a metric space (S, d).

(a) *The set E is closed if and only if $E = E^-$.*

(b) *The set E is closed if and only if it contains the limit of every convergent sequence of points in E.*

(c) *An element is in E^- if and only if it is the limit of some sequence of points in E.*

(d) *A point is in the boundary of E if and only if it belongs to the closure of both E and its complement.*

Pf: (a) $\Rightarrow$ $E^- \supset E$ by definition, and $E^- = \bigcap F$ for all

closed $F \supset E$, and we can take one of the $F = E$, hence

$E^- \subset E$. This shows $E^- = E$.

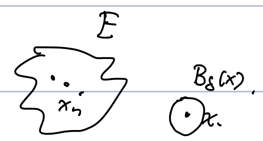
$\Leftarrow$ E^- is an intersection of closed sets, hence is

closed.

$x \subset h$.

(b): $\Rightarrow$. suppose E is closed. $x_n \to x$, $x_n \in E$, but $x \notin E$.

then $\exists \delta > 0$, s.t. $\underline{B_\delta(x) \cap E = \phi}$. $\Leftrightarrow B_\delta(x) \subset E^c$



this contradicts with $x_n \to x$.

$\Leftarrow$ suffice to prove that, if $\underline{x} \notin E$, then $\exists \delta > 0$, s.t.

$B_\delta(x) \subset E^c$. Suppose it's impossible to find such δ, i.e.

$\forall \delta > 0$, $B_\delta(x) \cap E \neq \phi$, then take δ to run through $\frac{1}{n}$:

let $x_n \in B_{\frac{1}{n}}(x) \cap E$. then $x_n \to x$ as a seq in S.

By assumption, $x \in E$. Hence we have contradiction with $x \notin E$.

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(c) $\Leftarrow$ if $\exists$ seq (p_n), s.t. $\lim p_n = p$ and $p_n \in E$ $\forall n$,

then by (b), $\forall F \supset E$ F closed, we have $\lim p_n = p$, $p_n \in F$

hence $p \in F$. Thus $p \in E^-$.

$\Rightarrow$ if $p \in E^-$, then for any $\varepsilon > 0$, $B_\varepsilon(p) \cap E \neq \phi$, otherwise.

E is contained in the closed set $B_\varepsilon(p)^c$, hence $E^- \subset B_\varepsilon(p)^c$

contradiction with $p \in E^-$, $p \notin B_\varepsilon(p)^c$. Hence, we may pick

points $p_n \in B_{\frac{1}{n}}(p) \cap E$, and (p_n) converge to p.

(d) Recall that $\partial E := E^- \setminus E^\circ$. Since

$E^\circ = \bigcup \{ \mathcal{U} \mid \mathcal{U} \subset E, \mathcal{U} \text{ open} \}$.

Hence $(E^\circ)^c = \bigcap \{ \mathcal{U}^c \mid \mathcal{U} \subset E, \mathcal{U} \text{ open} \}$

$= \bigcap \{ \mathcal{U}^c \mid \mathcal{U}^c \supset E^c, \mathcal{U} \text{ open} \}$

$= \bigcap \{ F \mid F \supset E^c, F \text{ closed} \} = (E^c)^-$

Hence $E^- \setminus E^\circ = E^- \cap (E^\circ)^c = E^- \cap (E^c)^-$.

13.7 Show that every open subset of $\mathbb{R}$ is the union of disjoint finite or infinite seq of open intervals.

Pf: Let $\mathcal{U} \subset \mathbb{R}$ be open.

(1) We first show that, $\forall p \in \mathcal{U}$, there is a maximal open interval $p \in (a(p), b(p)) \subset \mathcal{U}$. Indeed, by openness of $\mathcal{U}$, $\exists \delta > 0$, s.t. $(p-\delta, p+\delta) \subset \mathcal{U}$. Hence.

$\mathcal{U}^c = \mathcal{U}^c \cap (p-\delta, p+\delta)^c = (\mathcal{U}^c \cap (-\infty, p-\delta]) \cup (\mathcal{U}^c \cap (p+\delta, +\infty))$

$$=: \mathcal{U}_-^c \cup \mathcal{U}_+^c$$

Let $a(p) := \sup(\mathcal{U}_-^c)$, $b(p) := \inf(\mathcal{U}_+^c)$.

Since $\mathcal{U}_-^c$ is closed and bounded above, hence $a(p) \in \mathcal{U}_-^c$.

Similarly $b(p) \in \mathcal{U}_+^c$. $\forall q$, s.t. $a(p) < q < b(q)$,

since $q \notin \mathcal{U}_-^c$ and $q \notin \mathcal{U}_+^c$, thus $q \notin \mathcal{U}_-^c \cup \mathcal{U}_+^c = \mathcal{U}^c$.

Hence $q \in \mathcal{U}$. And since $a(p), b(p) \notin \mathcal{U}$, this interval is maximal.

(2). $\mathcal{U}$ is a disjoint union of such maximal intervals.

Indeed, every point of $\mathcal{U}$ is contained in such an interval and these intervals are disjoint from each other.

(3). Let A denote the collection of such intervals. Then for each $I \in A$, we may pick a rational point $P_I \in \mathbb{Q} \cap I$.

Since $\{P_I : I \in A\} \subset \mathbb{Q}$ is a subset of $\mathbb{Q}$ and biject with A, we have A is countable. Hence

$$\mathcal{U} = \bigcup_{I \in A} I.$$

is a disjoint union of countably many open.

including finite and infinitely countable