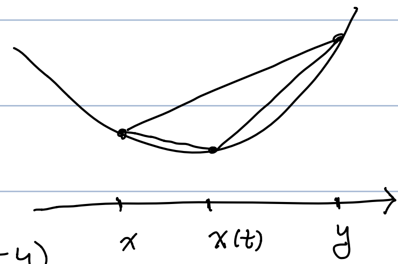
1. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that f is convex,

Show that $f'(x)$ is monotonously increasing.

Pf: For any real number $x < y$, we need to prove that $f'(x) \leq f'(y)$. Let $h = \dfrac{f(y) - f(x)}{y - x}$, we will show that $f'(x) \leq h$ and $h \leq f'(y)$.



Let $x(t) = (1-t) \cdot x + t \cdot y$, then.

Since $(1-t) f(x) + t \cdot f(y) \geq f((1-t)x + ty)$,

hence subtract both sides by $f(x)$, and divide by $x(t) - x = t(x-y)$,

we get

$$h \geq \frac{f(x(t)) - f(x)}{x(t) - x}$$

Let $t \to 0$, then the RHS $\to f'(x)$, hence

$$h \geq f'(x).$$

Similarly, $h \leq f'(y)$. Thus $f'(x) \leq h \leq f'(y)$. #

2. Let $f: (0, \infty) \to \mathbb{R}$ be a twice differentiable function.

Suppose $f(x) \to 0$ as $x \to \infty$, and $f''(x)$ is bounded.

Show that $f'(x) \to 0$ as $x \to 0$.

Pf: For any $x > 0$, $h > 0$, Using Taylor expansion of $f(x)$ at x, evaluated at $x+h$,

we have

$$f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2} \cdot f''(\xi)$$

for some $\xi \in (x, x+h)$.

Hence $f'(x) = \dfrac{f(x+h) - f(x)}{h} - \dfrac{h}{2} f''(\xi)$.

Fix an $\varepsilon > 0$. Then since $f(x) \to 0$ as $x \to \infty$, $\exists R > 0$. s.t. if $x > R$, then $|f(x)| < \varepsilon$. Also let $M > 0$, s.t. $|f''(t)| < M$ $\forall t > 0$. Then, $\forall x > R$, $h > 0$, we have

$$|f'(x)| \leqslant \frac{2\varepsilon}{h} + \frac{h}{2} M.$$

Since this inequality holds for all $h > 0$, we may pick an h to optimize the upper bound, i.e. make the RHS smallest.

Since $A + B \geqslant 2\sqrt{AB}$ and equality is achieved when $A = B$, thus $\frac{2\varepsilon}{h} + \frac{h}{2} M \geqslant 2\sqrt{\frac{2\varepsilon}{h} \cdot \frac{h}{2} M} = 2\sqrt{\varepsilon M}$, and is minimized when $h = 2\sqrt{\varepsilon / M}$. Hence

$$|f'(x)| \leqslant 2\sqrt{\varepsilon M} \qquad \forall x > R.$$

Thus for any $\varepsilon_0 > 0$, there exists an $\varepsilon > 0$, s.t. $2\sqrt{\varepsilon M} < \varepsilon_0$, and there is an $R > 0$, s.t. $\forall x > R$

$$|f'(x)| \leqslant 2\sqrt{\varepsilon M} < \varepsilon_0.$$

We have $\lim\limits_{x \to \infty} f'(x) = 0$

3. If $f: \mathbb{R} \to \mathbb{R}$ is a continuous function, such that $f'(x)$ exists for all $x \neq 0$. If we also know that $\lim\limits_{x \to 0} f'(x) = 5$. Show that $f'(0) = 5$.

Pf: We claim that $\lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h} = 5$, hence $f'(0) = 5$.

Indeed, using the mean value theorem applied to interval $[0,h]$ (assuming $h>0$ here, the $h<0$ case is similar). , then $\exists\, \xi \in (0,h)$ such that

$$\frac{f(h)-f(0)}{h} = f'(\xi)$$

Since $\lim_{x\to 0} f'(x) = 5$, hence $\forall\, \varepsilon>0$, $\exists\, \delta>0$. s.t. if $0<|x|<\delta$, then $|f'(x)-5|<\varepsilon$. Thus, if $0<h<\delta$, we have

$$\left|\frac{f(h)-f(0)}{h} - 5\right| = |f'(\xi)-5| < \varepsilon.$$

Hence $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = 5$. Similarly, $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = 5$.

Thus, $f'(0)=5$.

4. Give an example of bounded real function on $[0,1]$ that is not Riemann integrable.

Sol: Let $f(x) = \begin{cases} 1 & x\in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Then.

$U(P,f) = 1$, $L(P,f) = 0$. for all partitions.

Hence not integrable.