1. Let $f: [a,b] \to \mathbb{R}$ be differentiable. Prove that $f'(x)$ cannot have any simple discontinuities.

Pf: We prove by contradiction. Suppose there is a simple discontinuity at $x_0 \in [a,b]$, then $f'(x_0)$ is not equal to either the left or right limit of $f'(x)$ at $x = x_0$. WLOG, suppose $f'(x_0) \neq \lim_{x \to x_0^+} f'(x_0)$

Let $a = f'(x_0)$, $b = \lim_{x \to x_0^+} f'(x)$, and let $\varepsilon = |a-b|/2$.

By definition of limit, there exists $\delta > 0$, such that if $x \in (x_0, x_0+\delta)$, then $|f'(x) - b| < \varepsilon$. However, this contradict with IVT for derivative, applied to interval $[x_0, x_0+\delta/2]$.,

Since $f'(x_0) = a$, $|f'(x_0+\delta/2) - a| \geq |a-b| - |f'(x_0+\delta/2) - b| \geq \frac{|a-b|}{2}$.

hence for $\mu = \frac{2}{3}a + \frac{1}{3}b$ between a and b, there exists $f'(\gamma) = \mu$ $\gamma \in (x_0, x_0+\delta/2)$. This means

$$|f'(\gamma) - b| = |\mu - b| = \frac{2}{3}|a-b| > \frac{1}{2}|a-b| = \varepsilon.$$

contradict with the choice of δ.

2. If a sequence of differentiable function converges uniformly, does it mean $f(x)$ is differentiable?

Ans: No. ① For example, consider $f(x) = \max\{0, x\}$, $x \in \mathbb{R}$.

and $f_n(x) = \frac{1}{n}\log(1+e^{nx})$. Then each $f_n(x)$ is smooth and $f_n(x)$ converge uniformly to f. (adjust this example a bit to make the kink at $x=0$ for $f(x)$ lie in $[0,1]$).

② Another example is. Let $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x > 0 \\ 0 & x \leq 0 \end{cases}$

· Let $\varphi(x)$ be the smooth step function on $\mathbb{R}$, such that

$$\varphi(x) = \begin{cases} 0 & x \leq 0 \\ \in (0,1) & x \in (0,1) \\ 1 & x \geq 1 \end{cases}$$

· then we define

$$f_n(x) = f(x) \cdot \varphi(nx) .$$

Then

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} |f(x)| \cdot |\varphi(nx) - 1|$$

$$\leq \sup_{x \leq \frac{1}{n}} |f(x)| = \frac{1}{n}.$$

Hence $f_n(x) \to f(x)$ uniformly.

One can also check that $f_n(x)$ is smooth, especially.

$f_n^{(m)}(0) = 0, \ \forall m, n.$

3. Let $f(x) = x^4 . \left(2 + \sin\left(\frac{1}{x}\right)\right)$. Compute its derivative. and show that there is a sequence of local minima approaching $x = 0$.

if $x \neq 0$.

Pf: · $f'(x) = 4x^3 \left(2 + \sin\frac{1}{x}\right) + x^4 . \left(-\frac{1}{x^2} \cos\frac{1}{x}\right)$

$$= x^2 \left[-\cos\frac{1}{x} + 4x \cdot \left(2 + \sin\frac{1}{x}\right)\right].$$

· for $x = 0$, $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x^4 . [\sin(\frac{1}{x}) + 2]}{x} = 0.$

∴ Hence $f'(x)$ exists and is continuous. for all $x \in \mathbb{R}$.

- For $u>0$, let $g(u) = f(\frac{1}{u}) = u^{-4}(2+\sin u)$.
  Then $\{u>0 : g'(u)=0\}$ is in bijection with $\{x>0 : f'(x)=0\}$ under the map $u=\frac{1}{x}$. Moreover, local max / min of g and f are matched. Hence, suffice to prove that there is a sequence of local minimum of $g(u)$, as $u \to \infty$.

$$g'(u) = u^{-4}\left[-4\frac{(2+\sin u)}{u} + \cos u\right].$$

- Recall that, $\sin(u)$ has local min at $\left\{u : \begin{matrix} u>0 \\ (\sin u)'=0 \\ (\sin u)''>0 \end{matrix}\right\}$ (for $u>0$,)

$= \{u>0 : \cos u = 0,\ -\sin u > 0\} = \{2\pi n + \frac{3}{2}\pi : n=0,1,2,\cdots\}$.

- Let $C_n = 2\pi n + \frac{3}{2}\pi$. We will show that there exists $N>0$, s.t. $\forall n>N$, there exist a local minimum of $g(u)$ in $(C_n - \frac{\pi}{4}, C_n + \frac{\pi}{4})$.

- <u>Lemma</u>: If $h(u)$ is a smooth function on $u \in (0,\infty)$. and $h(u)$ and $h'(u)$ are bounded. Then for any $\varepsilon>0$, there is a $M>0$. s.t. $\forall u>M$. $|h(u)/u| < \varepsilon$, and $\left|\left(\frac{h(u)}{u}\right)'\right| < \varepsilon$.

Pf as exercise.

- Apply the Lemma to $h(u) = -4(2+\sin u)$, and $\varepsilon = \frac{1}{2}\cos(\frac{\pi}{4})$, then we get an $M>0$. Let N be the smallest integer. such that $C_N > M$.

• For any $n > N$, consider the interval $[c_n - \frac{\pi}{4}, c_n + \frac{\pi}{4}]$,

Let $G(u) = \cos(u) - \frac{4}{u}(2+\sin u)$, $H(u) = -\frac{4}{u}(2+\sin u)$.

$$G(c_n - \tfrac{\pi}{4}) = -\cos(\tfrac{\pi}{4}) + H(c_n - \tfrac{\pi}{4}) \leqslant -\cos(\tfrac{\pi}{4}) + \tfrac{1}{2}\cos(\tfrac{\pi}{4}) < 0$$

$$G(c_n + \tfrac{\pi}{4}) = \cos(\tfrac{\pi}{4}) + H(c_n + \tfrac{\pi}{4}) \geqslant \cos(\tfrac{\pi}{4}) - \tfrac{1}{2}\cos(\tfrac{\pi}{4}) > 0.$$

Hence there is a $r_n \in (c_n - \frac{\pi}{4}, c_n + \frac{\pi}{4})$, such that $G(r_n) = 0$.

• We next prove that $g''(u) > 0$ at $u = r_n$.

$$g''(u) = (u^{-4} \cdot G(u))' = -4 \cdot u^{-5} \cdot G(u) + u^{-4} \cdot G'(u).$$

Since at $u = r_n$, $G(u) = 0$, thus

$$g''(r_n) = r_n^{-4} \cdot G'(r_n).$$

$G'(u) = -\sin(u) + H'(u)$. For $u \in [c_n - \frac{\pi}{4}, c_n + \frac{\pi}{4}]$

$= [2\pi n + \frac{5}{4}\pi, 2\pi n + \frac{7}{4}\pi]$.

$\sin(u) \leqslant -\frac{\sqrt{2}}{2}$, $|H'(u)| \leqslant \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

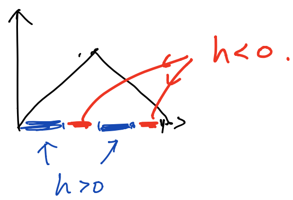
Hence $G'(u) \geqslant \frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} > 0$, thus $G'(r_n) > 0$.

This proves $g(u)$ has a sequence $\{r_n : n \geq N\}$ of local minum such that $r_n \in (2\pi n + \frac{5}{4}\pi, 2\pi n + \frac{7}{4}\pi)$, in particular $r_n \to \infty$.

#4: Prove that $f(x) = \sum_{n=0}^{\infty} 4^{-n} \varphi(4^n x)$ is a a continuous function that is nowhere differentiable, where $\varphi(x) = \min\{|x-n| : n \in \mathbb{Z}\}$

Pf: Let $h_0(x) = \begin{cases} \frac{1}{4} & x \in [n, n+\frac{1}{4}) \cup [n+\frac{1}{2}, n+\frac{3}{4}) \text{ for some } n \in \mathbb{Z} \\ -\frac{1}{4} & x \in [n+\frac{1}{4}, n+\frac{1}{2}) \cup [n+\frac{3}{4}, n+1), \text{ for some } n \in \mathbb{Z}. \end{cases}$

Then we can see that $\varphi_0(x)$ is monotonous on the interval between $x, x+h_0(x)$ $\forall x \in \mathbb{R}$.



Similarly, let $h_n(x) = 4^{-n} h_0(4^n x)$. Then we have,

If $m \leq n$, then

$$\frac{|\varphi_n(x+h_n(x)) - \varphi_n(x)|}{|h_n(x)|} = 1$$

and if $m > n$, then

$$\varphi_m(x+h_n(x)) - \varphi_m(x) = 0.$$

Hence $Q_n(x) = \dfrac{f(x+h_n(x)) - f(x)}{h_n(x)} = \displaystyle\sum_{m=0}^{n} \frac{\varphi_m(x+h_n(x)) - \varphi_m(x)}{h_n(x)}$

is a sum of ± 1 with $n+1$ entries. In particular, if n is even, then $Q_n(x)$ is odd; and if n is odd, then $Q_n(x)$ is even.

for each $x \in \mathbb{R}$,

Thus, consider the sequence $(x+h_n(x))_{n \in \mathbb{N}}$ approaching x, we see $\lim_{n\to\infty} Q_n(x+h_n(x))$ does not converge, since it is a sequence of integers with alternating oddity.