Math 104 Homework 1

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1.1 Proof by induction: Let $P(n)$ be "$1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$"

Base case: $1^2=\frac{1}{6}\times1\times2\times3=1$ $P(1)$ is true.

Suppose $P(k)$ is true for some $k\geq1$.

$$1^2+2^2+\cdots+(k+1)^2=1^2+\cdots+k^2+(k+1)^2=\frac{1}{6}k(k+1)(2k+1)+(k+1)^2$$
$$=\frac{(k+1)[(2k^2+k)+6(k+1)]}{6}$$
$$=\frac{1}{6}(k+1)(2k^2+7k+6)=\frac{1}{6}(k+1)(k+2)(2k+3)$$

Then $P(k+1)$ is true

Thus, $P(n)$ is true for all $n\in\mathbb{Z}^+$

1.3. Proof by induction: Let $P(n)$ be "$1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2$"

Base case: $1^3=1^2$ $P(1)$ is true.

Suppose $P(k)$ is true for some $k\geq1$

$$1^3+2^3+\cdots+k^3+(k+1)^3=(1+2+\cdots+k)^2+k^3+3k^2+3k+1$$
$$=(1+2+\cdots+k)^2+(k^2+2k+1)+(k^3+2k^2+k)$$
$$=(1+2+\cdots+k)^2+(k+1)^2+k(k+1)^2$$
$$=(1+2+\cdots+k)^2+(k+1)^2+2(k+1)(1+2+\cdots+k)$$
$$=[(1+2+\cdots+k)+(k+1)]^2$$

Then $P(k+1)$ is true.

Thus, $P(n)$ is true for all $n\in\mathbb{Z}^+$.

1.6. Proof by induction: Let $P(n)$ be "$7\mid 11^n-4^n$"

Base case: $11^1-4^1=7$ is divisible by 7. $P(1)$ is true.

Suppose $P(k)$ is true for some $k\geq1$

$$11^{k+1}-4^{k+1}=11\cdot11^k-4\cdot4^k=4\cdot11^k+7\cdot11^k-4\cdot4^k$$
$$=7\cdot11^k+4(11^k-4^k)$$

Since $7\mid(7\cdot11^k)$ and $7\mid(11^k-4^k)$, $7\mid[7\cdot11^k+4(11^k-4^k)]$

Then $P(k+1)$ is true

Thus, $P(n)$ is true for all $n\in\mathbb{Z}^+$.

2.5. $x=\sqrt{2+\sqrt{2}}$

$x^2=2+\sqrt{2}$

$(x^2-2)^2=2$

$\Rightarrow$ x is a root of $x^4-2x^2+2=0$

By corollary 2.3, the only rational numbers that can possibly be a solution to $x^4-2x^2+2=0$ are $\pm1, \pm2$.

When $x=1$, $1^4-2+2\neq0$ when $x=2$, $2^4-2\times4+2\neq0$

when $x=-1$, $(-1)^4-2+2\neq0$ when $x=-2$, $(-2)^4-2\times4+2\neq0$.

Thus, $x=\sqrt{2+\sqrt{2}}$ is not a rational number.

4.7. (a) By definition, $\inf S\leq s\leq \sup S$ for $\forall s\in S$

Then, by transitivity, $\inf S\leq \sup S$

By definition, $\inf T\leq t$, $t\leq \sup T$ for $\forall t\in T$

Since $S\subseteq T$, $\forall s\in S$, $s\in T$

Thus, $\inf T\leq s\leq \sup T$ for $\forall s\in S$.

Then, $\inf T$ is a lower bound of S. $\sup T$ is an upper bound of S.

Since $\inf S$ is the greatest lower bound, $\inf T\leq \inf S$. Similarly, $\sup S\leq \sup T$

Thus, $\inf T\leq \inf S\leq \sup S\leq \sup T$.

(b) WLOG, let $\sup S\leq \sup T$. Then $\max\{\sup S, \sup T\}=\sup T$

Since $s\leq \sup S$ for $\forall s\in S$, by transitivity, $s\leq \sup T$ for $\forall s\in S$

Also, $t\leq \sup T$ for $\forall t\in T$

Thus, $x\leq \sup T$ for $\forall x\in S\cup T$. $\sup T$ is an upperbound for $S\cup T$.

Suppose m is an upperbound for $S\cup T$, then $t\leq m$ for $\forall t\in T$

Then, m is an upper bound for T.

Since $\sup T$ is the least upper bound for T, $\sup T\leq m$

Thus, in this case, $\sup(S\cup T)=\sup T$

Similarly, if $\sup T\leq \sup S$, $\sup(S\cup T)=\sup S$

Thus, $\sup(S\cup T)=\max\{\sup S, \sup T\}$.

4.14 (a) First, we will show $\sup A + \sup B \leq \sup(A+B)$:

By definition, $a+b \leq \sup(A+B)$ for $\forall a \in A, \forall b \in B$

$a \leq \sup(A+B) - b$ for $\forall a \in A, \forall b \in B$

Then for $\forall b \in B$, $\sup(A+B) - b$ is an upper bound for A.

Hence, $\sup A \leq \sup(A+B) - b$ for $\forall b \in B$

$b \leq \sup(A+B) - \sup A$ for $\forall b \in B$

Then, $\sup(A+B) - \sup A$ is an upper bound for B

Hence $\sup B \leq \sup(A+B) - \sup A$

$\sup A + \sup B \leq \sup(A+B)$

Then, we will show $\sup(A+B) \leq \sup A + \sup B$:

By definition, $a \leq \sup A$ for $\forall a \in A$ $\Rightarrow a+b \leq \sup A + \sup B$ for $\forall a \in A, \forall b \in B$.

$b \leq \sup B$ for $\forall b \in B$. i.e. $\forall a+b \in A+B$.

Then, $\sup A + \sup B$ is an upper bound for $A+B$.

Hence, $\sup(A+B) \leq \sup A + \sup B$.

In conclusion, since $\left.\begin{array}{l}\sup A + \sup B \leq \sup(A+B) \\ \sup(A+B) \leq \sup A + \sup B\end{array}\right\} \Rightarrow \sup(A+B) = \sup A + \sup B$.

(b) First, we will show $\inf(A+B) \leq \inf A + \inf B$.

By definition, $a+b \geq \inf(A+B)$ for $\forall a \in A, \forall b \in B$.

$b \geq \inf(A+B) - a$ for $\forall a \in A, \forall b \in B$.

Then, $\forall a \in A$, $\inf(A+B) - a$ is a lower bound for B.

Hence $\inf(A+B) - a \leq \inf B$ for $\forall a \in A$

$a \geq \inf(A+B) - \inf B$ for $\forall a \in A$

Then, $\inf(A+B) - \inf B$ is a lower bound for A.

Hence $\inf(A+B) - \inf B \leq \inf A \Rightarrow \inf(A+B) \leq \inf A + \inf B$.

Then, we will show $\inf A + \inf B \leq \inf(A+B)$

By definition, $a \geq \inf A$ for $\forall a \in A$ $\Rightarrow a+b \geq \inf A + \inf B$ for $\forall a \in A, \forall b \in B$

$b \geq \inf B$ for $\forall b \in B$. i.e. $\forall a+b \in A+B$.

Then, $\inf A + \inf B$ is a lower bound for $A+B$.

Hence $\inf A + \inf B \leq \inf(A+B)$

In conclusion, $\left.\begin{array}{l}\inf(A+B) \leq \inf A + \inf B \\ \inf A + \inf B \leq \inf(A+B)\end{array}\right\} \Rightarrow \inf(A+B) = \inf A + \inf B$.

4.15. Suppose $a>b$ when $a \le b+\frac{1}{n}$ for $\forall n \in \mathbb{N}$

Since $a>b$, $a-b>0$.

By Archemedean property, $\exists n_0 \in \mathbb{N}$ such that $n_0(a-b)>1$

Then $a-b > \frac{1}{n_0}$

$a > b + \frac{1}{n_0}$

contradicts with $a \le b+\frac{1}{n}$ for $\forall n \in \mathbb{N}$.

Thus, $a \le b$.